

Nonlinear effects of atomic collisions on the optomechanical properties of a Bose-Einstein condensate in an optical cavity

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In this paper, we have investigated theoretically the influence of atomic collisions on the behaviour of a one-dimensional Bose-Einstein condensate inside a driven optical cavity. We develop the discrete-mode approximation for the condensate taking into account the interband transitions due to the s-wave scattering interaction. We show that in the Bogoliubov approximation the atom-atom interaction shifts the energies of the excited modes and also plays the role of an optical parametric amplifier for the Bogoliubov side mode which can affect its normal-mode splitting behaviour. On the other hand due to the atomic collisions the resonance frequency of the cavity is shifted which leads to the decrease of the number of cavity photons and the depletion of the Bogoliubov mode. Besides, it reduces the effective atom-photon coupling parameter which consequently leads to the decrease of the entanglement between the Bogoliubov mode and the optical field.

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I. INTRODUCTION

The emerging field of optomechanics concerns with the study of the mechanical effects of light on mesoscopic and macroscopic mechanical oscillators. This phenomenon has been realized in the optomechanical systems consisting of an optical cavity with a movable end-mirror or with a membrane-in-the middle. The radiation pressure exerted by the light inside the optical cavity couples the moving mirror or the membrane which acts as a mechanical oscillator to the optical field. This optomechanical coupling has been employed for a wide range of applications such as the cavity cooling of microlevers and nanomechanical resonators to their quantum mechanical ground state [1–5], producing high precision detectors for measuring weak forces and small displacements and also providing a good approach for fundamental studies of the transition between the quantum and the classical world [6–8].

On the other hand an alternative path to the studies of cavity optomechanics has been provided experimentally by systems consisting of ultracold atomic ensembles trapped in optical cavities [10–13] where the excitation of a collective mode of the cold gas plays the role of the vibrational mode of the mirror. The standing electromagnetic wave inside the cavity forms a periodic potential, the so-called optical lattice, in which the cold atoms exhibit phenomena typical of solid state physics like the formation of energy bands and Bloch oscillations [14].

In such systems with high finesse cavities the atom-light interaction is enhanced because the atoms are collectively coupled to the same optical mode. Besides, in

the dispersive regime where the laser pump is far detuned from the atomic resonance the excited electronic state of the atoms can be adiabatically eliminated and consequently the only degrees of freedom of atoms will be their mechanical motions [15–17]. For low photon numbers when the optical grating produced by the intracavity optical field is very shallow, one can approximately restrict the dynamics to the first two motional modes [24, 26]. In more recent theoretical investigations [23, 33] it has been shown that the simple two-mode model of a Bose-Einstein condensate (BEC) can be improved by considering higher motional modes.

In spite of similarities between the two kinds of optomechanical systems (with a moving mirror and with a BEC), there are some essential differences between them. Firstly, in contrast to the moving mirror of the optomechanical systems which is placed in the harmonic potential of its spring, the excitation modes of the BEC are not based on the presence of such an external harmonic potential [12]. Secondly, their parameters are realized in different regimes; the frequency of the excitation mode of the BEC, i.e., recoil frequency, is well below the dipole coupling strength while the oscillation frequency of a moving mirror is of the same order of magnitude as the coupling strength [13, 26]. On the other hand in addition to pumping the cavity from one of the end mirrors, in the cavities equipped with atomic gas it is also possible to pump the atoms from the side of the cavity [27].

One of the most important characteristics of the many-body systems is the two-body collision which can affect the properties of the system. So in order to study the dynamics of a BEC gas in a realistic experimental situation, it is necessary to take it into account. In the optomechanical systems containing the atomic gas there are two kinds of atom-atom interactions which are the

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origins of the system nonlinearities. Firstly, due to the atom-photon interaction the potential acting on the condensate depends in a highly nonlocal and nonlinear way on the condensate itself which leads to the long range atomic interaction mediated by the cavity field [28, 33]. On the other hand there is an intrinsic nonlinearity due to the s-wave scattering which can take place at arbitrary momentum values and causes a broadening of the atomic momentum distribution [29] due to the intraband transitions and also can scatter atoms to the other bands (interband transitions)[14, 20]. Furthermore, there is another kind of nonlinearity due to the finiteness of the particle number of the system which can be neglected in the thermodynamic limit where the total number of atoms is very large [40].

In this work, we are going to extend the two-mode model considering the effects of atom-atom interaction for a one-dimensional BEC in an optical cavity. We consider the system in the low photon regime but will not restrict our treatment to the weak atom-atom interaction. In fact the atomic collisions can populate several nonzero quasimomenta of the energy bands (intraband transition) and can also cause the atoms to be scattered to the other bands (interband transitions). In order to have a simplified optomechanical model we take into account just the lowest nonzero quasimomenta in the first Brillouin zone. We show that in the Bogoliubov approximation the atom-atom interaction not only shifts the energies of the excited modes but also play the role of an optical parametric amplifier (OPA) for the Bogoliubov side mode which can affect its normal-mode splitting behaviour [9]. On the other hand due to the atomic collisions the resonance frequency of the cavity is shifted which leads to the decrease of the number of cavity photons and the depletion of the Bogoliubov mode. These results are in good agreement with those obtained by the numerical solutions of the full description of Gross-Pitaevskii Equation (GPE) [33, 34].

The paper is structured as follows. In Sec. II we will first give a thorough theoretical description of the many-body system under consideration and then derive a simplified optomechanical model. In Sec. III the quantum Langevin equations (QLEs) are derived and linearized around the semiclassical steady state. In Sec. IV we study the mean-field solutions and fluctuations of the system and in Sec. V we derive the spectrum of the Bogoliubov mode and investigate the normal mode splitting (NMS). Finally, our conclusions are summarised in Sec. VI.

II. THEORETICAL DESCRIPTION OF THE SYSTEM

We are going to study a gas of N ultracold bosonic two-level atoms with mass M and transition frequency ω_a inside the optical lattice of a single-mode, high-finesse Fabry-Perot cavity with length L . The cavity is driven at

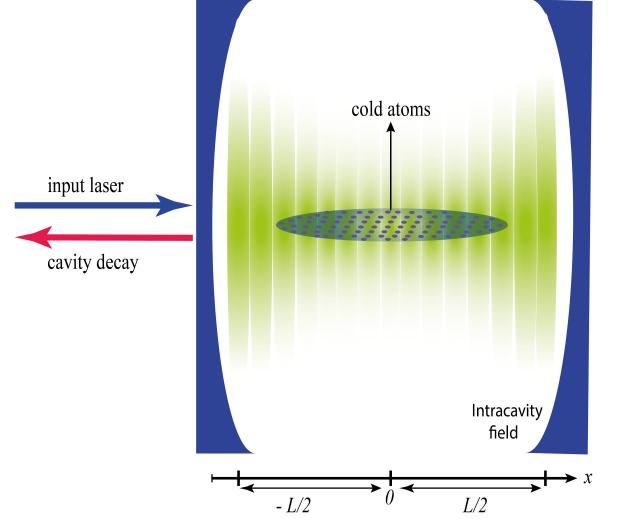


FIG. 1. (Color online) N two-level atoms trapped in an optical cavity interacting dispersively with a single cavity mode. The cavity mode is driven by a laser at rate η and the decay rate is κ .

rate η through one of its mirrors by a laser with frequency ω_p , and wavenumber $K = 2\pi/\lambda = \omega_p/c$. We assume the BEC is confined in a cylindrically symmetric trap with a transverse trapping frequency ω_\perp and negligible longitudinal confinement along the x direction (Fig.1). In this way we can describe the dynamics within an effective one-dimensional model by quantizing the atomic motional degree of freedom along the x axis only.

A. The General Form of The Hamiltonian of a BEC Inside an Optical Lattice

In the dispersive regime where the laser pump is far detuned from the atomic resonance ($\Delta_a = \omega_p - \omega_a$ exceeds the atomic linewidth γ by orders of magnitude), the excited electronic state of the atoms can be adiabatically eliminated and spontaneous emission can be neglected [15]. In the frame rotating at the pump frequency, the many-body Hamiltonian reads

$$H = -\hbar\Delta_c a^\dagger a + i\hbar\eta(a^\dagger - a) + \int_{-L/2}^{L/2} dx \Psi^\dagger(x) H_0 \Psi(x) + H_{aa}, \quad (1)$$

where a is the annihilation operator for a cavity photon and $\Delta_c = \omega_p - \omega_c$ is the cavity-pump detuning. H_0 is the single-particle Hamiltonian of an atom inside the optical lattice of the cavity and H_{aa} is the atom-atom interaction that are respectively given by

$$H_0 = p^2/2M + \hbar U_0 \cos^2(Kx) a^\dagger a, \quad (2a)$$

$$H_{aa} = \frac{1}{2} U_s \int_{-L/2}^{L/2} dx \Psi^\dagger(x) \Psi^\dagger(x) \Psi(x) \Psi(x). \quad (2b)$$

Here $U_0 = g_0^2/\Delta_a$ is the optical lattice barrier height per photon and represents the atomic backaction on the field, g_0 is the vacuum Rabi frequency, $U_s = \frac{4\pi\hbar^2 a_s}{M}$ and a_s is the two-body s-wave scattering length [16, 17]. The second term in the Hamiltonian of Eq.(2a) is a periodic potential of period $\lambda/2$.

It is well-known that the eigenfunctions of a particle inside a periodic potential are the Bloch functions $\psi_{\nu,q}(x)$ with eigenvalues $\epsilon_{\nu,q}$ where ν is the Bloch band index and $q \in [-2\pi/\lambda, 2\pi/\lambda]$ is the quasimomentum of the particle [18]. If there are l periods inside the cavity, then $L = l\lambda/2$. Using the Born-Von Karman periodic boundary condition $\psi_{\nu,q}(x + L) = \psi_{\nu,q}(x)$ and based on the Bloch theorem,

$$\psi_{\nu,q}(x + l\lambda/2) = e^{iql\lambda/2} \psi_{\nu,q}(x), \quad (3)$$

it is deduced that $ql\lambda/2 = 2m\pi$, where ($m \in \mathbb{Z}$). In this way the quasimomentum q_m in the first Brillouin zone can be written in terms of optical wave number, i.e., $q_m = 2mK/l$ where $-l/2 \leq m \leq l/2$. Now using the Bloch theorem, the eigenfunctions $\psi_{\nu,q_m}(x)$ can be written as

$$\psi_{\nu,q}(x) = \frac{1}{\sqrt{l}} e^{iq_m x} u_{\nu}(x). \quad (4)$$

Here $u_{\nu}(x)$ is a periodic function with the optical lattice period ($\lambda/2$) and reads

$$u_{\nu}(x) = \frac{1}{\sqrt{\lambda/2}} \sum_n c_{\nu,n} e^{i2nKx}. \quad (5)$$

By substituting Eq.(5) into Eq.(4), the eigenfunctions of the single-particle Hamiltonian H_0 is obtained as

$$\psi_{\nu,q_m}(x) = \frac{1}{\sqrt{L}} \sum_n c_{\nu,n} e^{i(q_m + 2nK)x}. \quad (6)$$

The second quantized atomic wave field $\Psi(x)$ can be expanded as a Fourier series in terms of the Bloch functions, i.e.,

$$\Psi(x) = \sum_{\nu,q_m} \tilde{b}_{\nu,q_m} \psi_{\nu,q_m}(x). \quad (7)$$

Here \tilde{b}_{ν,q_m} ($\tilde{b}_{\nu,q_m}^{\dagger}$) is the annihilation (creation) operator for the atomic field that annihilates (creates) a particle in a state determined with the Bloch band index ν and quasimomentum q_m . By using this expansion the Hamiltonian of the system is diagonalized and one can investigate the Bloch band structure of the condensate [19]. However, the Fourier coefficients $c_{\nu,n}$ can only be determined numerically and there is no closed form for the Bloch functions. Instead, by substituting Eq.(6) into Eq.(7) one can obtain a Fourier expansion of the atomic field in terms of plane waves

$$\Psi(x) = \frac{1}{\sqrt{L}} \sum_{n,m} b_{n,m} e^{i2K(n+m/l)x}. \quad (8)$$

In the derivation of Eq.(8) we have done a Bogoliubov transformation

$$b_{n,m} = \sum_{\nu} c_{\nu,n} \tilde{b}_{\nu,q_m}, \quad (9)$$

where $b_{n,m}$ ($b_{n,m}^{\dagger}$) is the atomic field operator that annihilates (creates) a particle in a state determined with the band index n and quasimomentum q_m .

By substituting Eq.(8) into Eq.(1) the Hamiltonian of the systems can be written as

$$\begin{aligned} H = & -\hbar\Delta_c a^{\dagger}a + i\hbar\eta(a^{\dagger} - a) + \sum_{n,m} \varepsilon_{nm} b_{n,m}^{\dagger} b_{n,m} \\ & + \frac{1}{4} \hbar U_0 a^{\dagger}a \sum_{n,m} b_{nm}^{\dagger} (b_{n-1,m} + b_{n+1,m} + 2b_{n,m}) \\ & + H_{aa}. \end{aligned} \quad (10)$$

In these sums $n \in \mathbb{Z}$, $-l/2 \leq m \leq l/2$, and

$$\varepsilon_{n,m} = 4E_R \left(n + \frac{m}{l} \right)^2 \quad (11)$$

are the energy eigenvalues in which $E_R = \hbar^2 K^2 / 2M$ is the recoil energy. As seen, the field expansion of Eq.(8) can no longer diagonalize the Hamiltonian of the system. In fact that part of Hamiltonian which corresponds to the atom-photon interaction remains nondiagonal. Due to the interaction with the optical field, the atoms are scattered to the nearest bands ($\Delta n = \pm 1$) while their quasimomentum (m) remains unchanged.

B. Optomechanical Model

Since the optical potential is symmetric with respect to the origin, the Hamiltonian of the system has the parity symmetry. So if the system starts from a homogeneous BEC, only the cosine parts of the exponential functions in Eq.(8) are excited because of the parity conservation. In the limit of weak photon-atom coupling, when either the photon number inside the cavity or U_0 is small, the lowest bands $n = \pm 1$ can be excited by fluctuations resulting from the atom-light interaction [26]. On the other hand, the s-wave scattering populates fluctuations with arbitrary n and m [20]. In a very simplified optomechanical model one can consider the scattering from $m = 0$ to $m = \pm 1$ in the lowest band $n = 0$. In this way the atomic field operator (Eq.(8)) can be truncated as

$$\Psi(x) = \frac{1}{\sqrt{L}} c_{00} + \sqrt{\frac{2}{L}} c_{10} \cos(2Kx) + \sqrt{\frac{2}{L}} c_{01} \cos(2Kx/l), \quad (12)$$

where we have done the following Bogoliubov transformations

$$c_{nm} = \frac{1}{\sqrt{2}} (b_{nm} + b_{-n,-m}), \quad (13a)$$

$$c_{n,-m} = \frac{1}{\sqrt{2}} (b_{n,-m} + b_{-n,m}). \quad (13b)$$

In the case that the system does not have parity symmetry, for example when the BEC is inside a ring cavity, one should also consider sine modes with annihilation operators

$$s_{nm} = \frac{1}{\sqrt{2}}(b_{nm} - b_{-n,-m}), \quad (14)$$

which in our model have been set aside [21, 22]. By substituting the atomic field operator, Eq.(12), into the Hamiltonian of Eq.(1), one can arrive at the following Hamiltonian

$$H = -\hbar\Delta_c a^\dagger a + i\hbar\eta(a - a^\dagger) + 4E_R(c_{10}^\dagger c_{10} + \frac{1}{l^2}c_{01}^\dagger c_{01}) + H_{ac} + H_{aa}, \quad (15)$$

where

$$H_{ac} = \frac{1}{2}\hbar U_0 a^\dagger a(c_{00}^\dagger c_{00} + c_{10}^\dagger c_{10} + c_{01}^\dagger c_{01} + \frac{1}{\sqrt{2}}c_{00}^\dagger c_{10} + \frac{1}{\sqrt{2}}c_{00} c_{10}^\dagger), \quad (16a)$$

$$H_{aa} = \hbar\frac{\omega_{sw}}{4N}(c_{00}^\dagger c_{10}^2 + c_{01}^2) + c_{00}^2(c_{10}^{\dagger 2} + c_{01}^{\dagger 2}) + 4c_{00}^\dagger c_{00} c_{10}^\dagger c_{10} + 4c_{00}^\dagger c_{00} c_{01}^\dagger c_{01} + c_{10}^{\dagger 2} c_{01}^2 + c_{10}^2 c_{01}^{\dagger 2} + 4c_{10}^\dagger c_{10} c_{01}^\dagger c_{01} + \frac{3}{2}c_{01}^{\dagger 2} c_{01}^2 + \frac{3}{2}c_{10}^{\dagger 2} c_{10}^2 + c_{00}^{\dagger 2} c_{00}^2). \quad (16b)$$

The first two terms in Eq.(15) denote the cavity and the pump Hamiltonians. The third term is the energy of the side-modes c_{10} and c_{01} which is just the third term in Eq.(10) with energy eigenvalues ε_{nm} given by Eq.(11). The Hamiltonian H_{ac} denotes the atom-photon interaction and H_{aa} is the atom-atom interaction. Furthermore, $\omega_{sw} = 8\pi\hbar a_s N / M l w^2$ is the s-wave scattering frequency and w is the waist of the optical potential.

This Hamiltonian can be further simplified since for weak optical fields and large N the depletion of the initial condensate remains weak. So we can use the Bogoliubov approximation and treat the zero-momentum mode classically, i.e., $c_{00} \rightarrow \sqrt{N}$ [24, 29]. In this way Eqs.(15) reduce to the following forms

$$H = -\hbar\tilde{\Delta}_c a^\dagger a + i\hbar\eta(a - a^\dagger) + \hbar(4\omega_R + \omega_{sw})c_{10}^\dagger c_{10} + \hbar(\frac{4}{l^2}\omega_R + \omega_{sw})c_{01}^\dagger c_{01} + H_{ac} + H_{aa}, \quad (17a)$$

$$H_{ac} = \frac{\sqrt{2N}}{4}\hbar U_0 a^\dagger a(c_{10} + c_{10}^\dagger) + \frac{1}{2}\hbar U_0 a^\dagger a(c_{10}^\dagger c_{10} + c_{01}^\dagger c_{01}), \quad (17b)$$

$$H_{aa} = \frac{1}{4}\hbar\omega_{sw}(c_{10}^2 + c_{10}^{\dagger 2} + c_{01}^2 + c_{01}^{\dagger 2}). \quad (17c)$$

Here $\tilde{\Delta}_c = \Delta_c - NU_0/2$ is the effective Stark-shifted detuning. As seen from the Hamiltonian of Eq.(17b) there are two kinds of optomechanical coupling. The first term

of Eq.(17b) is the linear radiation pressure which couples the side mode c_{10} to the optical field with the optomechanical coupling constant $\frac{\sqrt{2N}}{4}\hbar U_0$ while the second term is the nonlinear optomechanical coupling of the two side modes c_{10} and c_{01} with the optical field. In the atom-atom interaction Hamiltonian H_{aa} we have disregarded all terms proportional to ω_{sw}/N . The influence of s-wave scattering has partly appeared as a shift in the side modes energies (the third and fourth terms in Eq.(17a)). Since $\hbar\omega_{sw} = 2U_s n_A$, where $n_A = N/l w^2$ is the density of atoms, it is apparent that the energy shifts obtained here is the same as that of Bogoliubov theory [25]. On the other hand the role of H_{aa} in Eq.(17) for the Bogoliubov side modes is very similar to that of an optical parametric amplifier (OPA) in an optomechanical system with the nonlinear gain parameter ω_{sw} [9]. In Sec. V we will show how this parameter can affect the NMS behaviour of the coupled Bogoliubov mode and the cavity field.

III. DYNAMICS OF THE OPTOMECHANICAL SYSTEM

The dynamics of the optomechanical system described by Eqs.(17) is fully characterized by the following set of nonlinear QLEs, written in the frame rotating at the input laser frequency,

$$\dot{a} = (i\tilde{\Delta}_c - \kappa)a - \frac{iU_0}{2}a[\frac{\sqrt{2N}}{2}(c_{10} + c_{10}^\dagger) + c_{10}^\dagger c_{10}] - \eta + \xi, \quad (18a)$$

$$\dot{c}_{10} = -(i\omega_{10} + \gamma)c_{10} - i\frac{\sqrt{2N}}{4}U_0 a^\dagger a - \frac{iU_0}{2}a^\dagger a c_{10} - \frac{i}{2}\omega_{sw}c_{10}^\dagger + f_{10}, \quad (18b)$$

$$\dot{c}_{01} = -(i\omega_{01} + \gamma)c_{01} - \frac{iU_0}{2}a^\dagger a c_{01} - \frac{i}{2}\omega_{sw}c_{01}^\dagger + f_{01}, \quad (18c)$$

where $\omega_{10} = 4\omega_R + \omega_{sw}$ and $\omega_{01} = \frac{4}{l^2}\omega_R + \omega_{sw}$. Here κ and γ characterize the dissipation of the cavity field and collective density excitations of the BEC, respectively. The cavity-field quantum vacuum fluctuation $\xi(t)$ satisfies the Markovian correlation functions, i.e., $\langle \xi(t)\xi^\dagger(t) \rangle = (n_{ph} + 1)\delta(t - t')$, $\langle \xi^\dagger(t')\xi(t) \rangle = n_{ph}\delta(t - t')$ with the average thermal photon number n_{ph} which is nearly zero at optical frequencies [31]. Besides, $f_{10}(t)$ and $f_{01}(t)$ are the thermal noise inputs for the two side modes of BEC which also satisfy the same Markovian correlation functions as those of the optical noise. The noise sources are assumed uncorrelated for the different modes of both the matter and light fields.

A. Linearization

Now we are going to study the weak excitations of the condensate from its ground state. Such excitations include small deviations of both the atomic wave function and the optical field from their respective stationary states. So we decompose each operator in Eqs.(18) as the sum of its steady-state value and a small fluctuation. By substituting $a = \alpha + \delta a$, $c_{10} = \sqrt{N}\beta_1 + \delta c_{10}$ and $c_{01} = \sqrt{N}\beta_0 + \delta c_{01}$ into Eqs.(18) one can obtain a set of nonlinear algebraic equations for the steady-state values,

$$\alpha = \frac{\eta}{\sqrt{\Delta_d^2 + \kappa^2}}, \quad (19a)$$

$$\beta_1 = \frac{\sqrt{2}}{4} \frac{U_0 \alpha^2}{\sqrt{\Omega_{10}^{(+)} + \gamma^2}}, \quad (19b)$$

$$\beta_0 = 0, \quad (19c)$$

where we have assumed α, β_0 and β_1 are real numbers [23] and $\Delta_d = \tilde{\Delta}_c - \frac{1}{2}NU_0\beta_1(\beta_1 + \sqrt{2})$ is the effective detuning, $\Omega_{10}^{(\pm)} = \tilde{\omega}_{10} \pm \frac{1}{2}\omega_{sw}$ and $\tilde{\omega}_{10} = \omega_{10} + \frac{1}{2}U_0\alpha^2$. Eq.(19c) shows that the mean value of the mode c_{01} is zero. It is the consequence of the fact that the ground state of the translationally invariant system is also invariant under discrete translation [20]. On the other hand, the linearized QLEs for the fluctuating operators take the followinf forms

$$\delta\dot{a} = (i\Delta_d - \kappa)\delta a - \frac{i}{2}G(\delta c_{10}^\dagger + \delta c_{10}) + \xi, \quad (20a)$$

$$\begin{aligned} \delta\dot{c}_{10} = & -(i\tilde{\omega}_{10} + \gamma)\delta c_{10} - \frac{i}{2}G(\delta a + \delta a^\dagger) \\ & - \frac{i}{2}\omega_{sw}\delta c_{10}^\dagger + f_{10}, \end{aligned} \quad (20b)$$

$$\delta\dot{c}_{01} = -(i\tilde{\omega}_{01} + \gamma)\delta c_{01} - \frac{i}{2}\omega_{sw}\delta c_{01}^\dagger + f_{01}, \quad (20c)$$

where

$$G = U_0\sqrt{N}\alpha(\beta_1 + \frac{\sqrt{2}}{2}) \quad (21)$$

is the effective atom-photon coupling parameter. By defining the quadrature fluctuations

$$\delta X_a = \frac{1}{\sqrt{2}}(\delta a + \delta a^\dagger), \quad \delta P_a = \frac{1}{\sqrt{2}i}(\delta a - \delta a^\dagger), \quad (22a)$$

$$\delta X_j = \frac{1}{\sqrt{2}}(\delta c_j + \delta c_j^\dagger), \quad \delta P_j = \frac{1}{\sqrt{2}i}(\delta c_j + \delta c_j^\dagger), \quad (22b)$$

with $j = (10, 01)$, the QLEs can be written in the compact matrix form

$$\dot{u}(t) = Au(t) + n(t), \quad (23)$$

where $u = [\delta X_a, \delta P_a, \delta X_{10}, \delta P_{10}, \delta X_{01}, \delta P_{01}]^T$ is the vector of continuous variable fluctuation operators and

$$n(t) = [\xi_x(t), \xi_p(t), f_{x10}(t), f_{p10}(t), f_{x01}(t), f_{p01}(t)]^T. \quad (24)$$

is the corresponding vector of noises. The 6×6 matrix A is the drift matrix given by

$$A = \begin{pmatrix} -\kappa & -\Delta_d & 0 & 0 & 0 & 0 \\ \Delta_d & -\kappa & -G & 0 & 0 & 0 \\ 0 & 0 & -\gamma & \Omega_{10}^{(-)} & 0 & 0 \\ -G & 0 & -\Omega_{10}^{(+)} & -\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & -\gamma & \Omega_{01}^{(-)} \\ 0 & 0 & 0 & 0 & -\Omega_{01}^{(+)} & -\gamma \end{pmatrix}. \quad (25)$$

As is seen from the drift matrix, the side mode $(\delta X_{01}, \delta P_{01})$ has been decoupled from the optical mode. It is due to the fact that in the QLEs, Eqs.(18c), there is no standard radiation pressure coupling between the side mode c_{01} and the optical field. Instead, it is coupled to the radiation field via the nonlinear term, $-\frac{iU_0}{2}a^\dagger a c_{01}$ which is as small as $1/\sqrt{N}$ of the radiation pressure term, and is deleted during the linearization process (because the mean field value of this mode, β_0 , is zero). In this way the side mode c_{01} acts as a medium level in the dynamics.

B. Stationary Quantum Fuctuations

In order to study the stationary properties of the system it is enough to focus our attention on the subspace spanned by the optical mode and the side mode c_{10} . It means that we can consider only the upper block of the drift matrix of Eq.(25). The system is stable only if the real part of all the eigenvalues of the matrix A are negative. These stability conditions can be obtained by using the Routh-Hurwitz criterion [30]. Due to the linearized dynamics of the fluctuations and since all noises are Gaussian the steady state is a zero-mean Gaussian state which is fully characterized by the 4×4 stationary correlation matrix (CM) V , with components $V_{ij} = \langle u_i(\infty)u_j(\infty) + u_j(\infty)u_i(\infty) \rangle / 2$. When the system is stable such a CM is given by [32]

$$V_{ij} = \sum_{k,l} \int_0^\infty ds \int_0^\infty ds' M_{ik}(s) M_{jl}(s') D_{kl}(s-s'), \quad (26)$$

where $M(s) = \exp(As)$ and $D_{kl}(s-s') = \langle n_k(s)n_l(s') + n_l(s')n_k(s) \rangle / 2$ is the matrix of stationary noise correlation functions. For the noise diffusion matrix we have $D_{kl}(s-s') = D_{kl}\delta(s-s')$, where $D_{kl} = \text{Diag}[\kappa, \kappa, \gamma(2n_B+1), \gamma(2n_B+1)]$ is the diffusion matrix, with $n_B = [\exp(\hbar\omega_m/k_B T) - 1]^{-1}$ as the mean number of thermal excitations of the Bogoliubov side mode $(\delta X_{10}, \delta P_{10})$ whose frequency of oscillations is given by

$$\omega_m = \sqrt{\Omega_{10}^{(+)}\Omega_{10}^{(-)}}. \quad (27)$$

Therefore, Eq.(26) becomes

$$V = \int_0^\infty ds M(s) D M^T(s). \quad (28)$$

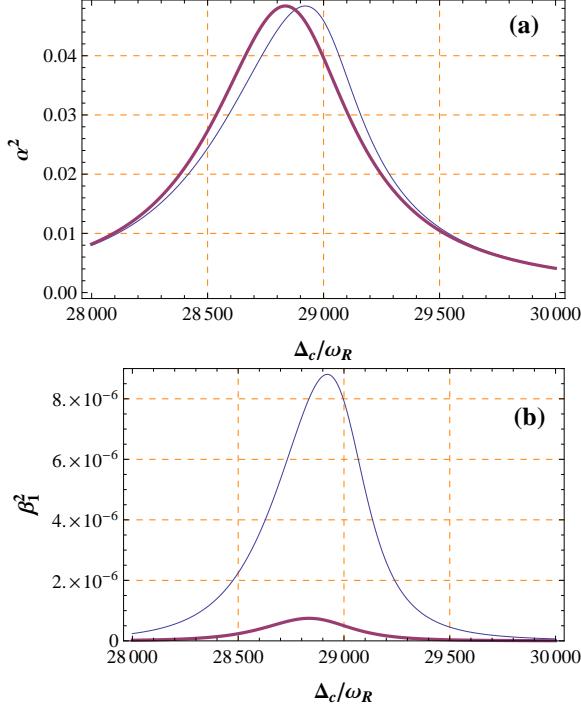


FIG. 2. (Color online) (a) The mean cavity photon number and (b) the mean value fraction of atoms in the Bogoliubov side mode c_{10} versus the normalized cavity detuning Δ_c/ω_R for two values of $\omega_{sw} = \omega_R$ (thin line) and $\omega_{sw} = 10\omega_R$ (thick line). The parameters are $N = 6 \times 10^4$, $U_0 = 0.96\omega_R$, $\kappa = 363.9\omega_R$, $\gamma = 0.001\kappa$, $\eta = 80.06\omega_R$ and $T = 10^{-7} K$.

When the stability conditions are satisfied $M(\infty) = 0$ and Eq. (28) will be equivalent to the following Lyapunov equation for the steady-state CM

$$AV + VA^T = -D. \quad (29)$$

Equation(29) is linear in V and can be straightforwardly solved. However, the explicit form of V is complicate and is not reported here.

IV. NUMERICAL SOLUTIONS TO THE MEAN-FIELDS AND FLUCTUATIONS

In this section we first discuss our results based on the numerical solutions of Eqs.(19) for the mean fields and then solve Eq.(29) to obtain fluctuations in the number of atoms and photons and their entanglement. We will show how the atom-atom interaction affects the mean fields, cavity resonance and atom-photon entanglement. We analyse our results based on the experimentally feasible parameters given in Ref.[12, 13].

A. The mean-field solution

Here we are going to study the effect of s-wave scattering on the behaviour of the mean-field values of the

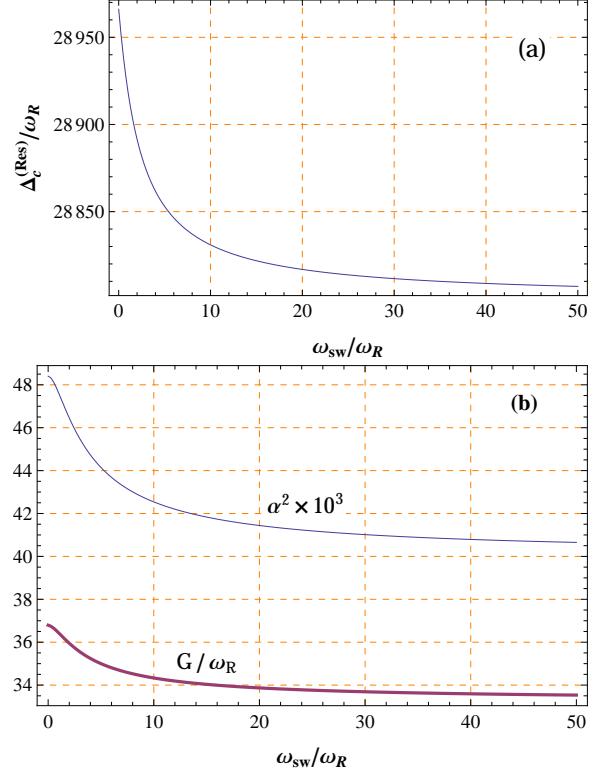


FIG. 3. (Color online) (a) The normalized resonance frequency of the cavity $\Delta_c^{(Res)}/\omega_R$ and (b) the normalized effective atom-photon coupling G/ω_R (thick line) and the mean photon number magnified by 10^3 (thin line) versus the normalized s-wave scattering frequency ω_{sw}/ω_R . The parameters are the same as those of Fig.2. The cavity detuning has been set at $\Delta_c = 28966\omega_R$.

optical field and the Bogoliubov side mode c_{10} , given by Eqs.(19), quantitatively. We consider $N = 6 \times 10^4$ atoms distributed in the optical cavity of length $L=178$ μm with bare frequency ω_c corresponding to a wavelength of $\lambda=780$ nm. The optical mode is coherently driven at rate $\eta=80.06\omega_R$ with the recoil frequency for rubidium atoms $\omega_R = 2\pi \times 3.57$ kHz and the one-atom light shift $U_0=0.96\omega_R$ [12, 20].

In Fig.2 we have plotted the mean number of photons and the fraction of condensate atoms occupying the Bogoliubov mode c_{10} versus the normalized atomic detuning Δ_c/ω_R at a fixed temperature of $T = 10^{-7}$ K, and for two values of s-wave scattering frequencies ω_{sw} . As is seen the increasing of the s-wave scattering interaction shifts the resonance frequency of the cavity to the lower values. Besides, it causes the number of atoms in the Bogoliubov mode to decrease i.e., leads to the depletion of this mode. This is in good accordance with the results obtained from numerical solution of the GPE (Fig.1 of Ref.[33]).

From Eq.(19 a) one can obtain the resonance condition

as

$$\Delta_c^{(Res)} = \frac{NU_0}{2}(1 + \sqrt{2}\beta_1 + \beta_1^2). \quad (30)$$

Such a resonance shift has been obtained in Ref.[13] without considering atomic collisions. As was mentioned in that reference the BEC acts as a Kerr medium that shifts the empty-cavity resonance. However, due to the atomic collisions the Bogoliubov side mode c_{10} is depleted to other modes i.e., the mean value β_1 decreases. So a BEC with atom-atom interaction has a different resonance in comparison to a non-interacting one. For clarity, we have shown the effect of atom-atom interaction on the resonance of the cavity in Fig.3(a). For a non interacting BEC, i.e., $\omega_{sw} = 0$, the resonance occurs at $\Delta_c^{(Rs)} = 28966 \omega_R$ while for very large values of ω_{sw} it goes to $NU_0/2 = 28800\omega_R$. In Fig.3(b) the effective atom-photon coupling parameter G (thick line), given by Eq.(21), and the mean photon number of the cavity (thin line) have been plotted versus the normalized s-wave scattering frequency. This figure shows that both the effective atom-photon coupling parameter and the mean number of photons decrease with ω_{sw} . All of these reductions are the direct consequence of the resonance shift exerted by atomic collisions. In the next subsection it will be shown that increasing the strength of atomic interaction causes the number of fluctuating photons to decrease. So the decrease of the total cavity photon number is a sign of atomic collisions. These results coincide with those obtained by numerical solution of GPE [34].

B. Fluctuations and Entanglement

After calculating the mean-field values we can obtain the elements of the drift matrix A and solve for the steady solutions of Eq.(23). As explained before, by solving the Lyapunov equation (Eq.(29)) we can obtain the correlation matrix V which gives us the second-order correlations of the fluctuations. The correlation matrix corresponding to the upper block of A in Eq.(25) can be written as

$$V = \begin{pmatrix} \mathcal{A} & \mathcal{C} \\ \mathcal{C}^T & \mathcal{B} \end{pmatrix}, \quad (31)$$

where \mathcal{A} and \mathcal{B} represent the correlations of the photonic and atomic degrees of freedom respectively, and \mathcal{C} describes the cross correlations. In this way we can calculate the incoherent excitation number of photons

$$\delta n_{ph} = \langle \delta a^\dagger \delta a \rangle = \frac{V_{11} + V_{22} - 1}{2}, \quad (32)$$

and the incoherent excitation number of atoms in the Bogoliubov side mode

$$\delta n_B = \langle \delta c_{10}^\dagger \delta c_{10} \rangle = \frac{V_{33} + V_{44} - 1}{2}. \quad (33)$$

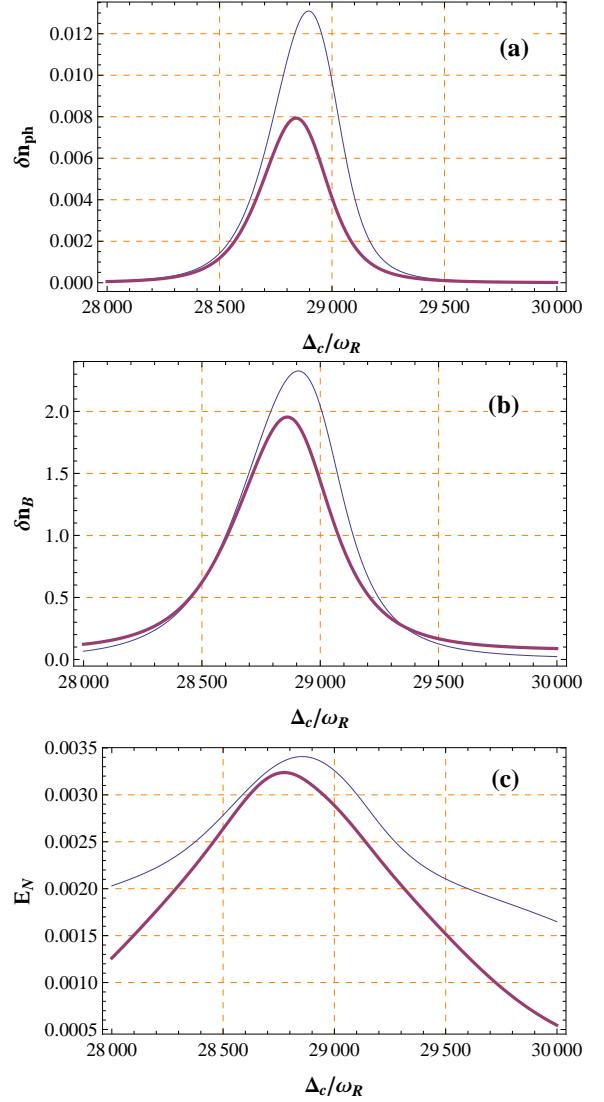


FIG. 4. (Color online) The incoherent excitation numbers of the photons (a), the incoherent excitation numbers of atoms in the Bogoliubov side mode c_{10} (b) and the entanglement between the Bogoliubov mode c_{10} and the optical field (c) versus normalized cavity detuning Δ_c/ω_R for two different values of $\omega_{sw} = \omega_R$ (thin line) and $\omega_{sw} = 10\omega_R$ (thick line). All parameters are the same as those of Fig.2.

On the other hand the bipartite entanglement between the atomic and photonic degrees of freedom can also be calculated by using the logarithmic negativity [35]

$$E_N = \max[0, -\ln 2\eta^-], \quad (34)$$

where $\eta^- \equiv 2^{-1/2} \left[\Sigma(V) - \sqrt{\Sigma(V)^2 - 4\det V} \right]^{1/2}$ is the lowest symplectic eigenvalue of the partial transpose of the 4×4 CM, V , with $\Sigma(V) = \det \mathcal{A} + \det \mathcal{B} - 2\det \mathcal{C}$.

In Figs.4 (a) and (b) we have plotted, respectively, the incoherent excitation number of photons and atoms versus the normalized cavity detuning Δ_c/ω_R for two values of ω_{sw} . As is seen, fluctuations are small far from

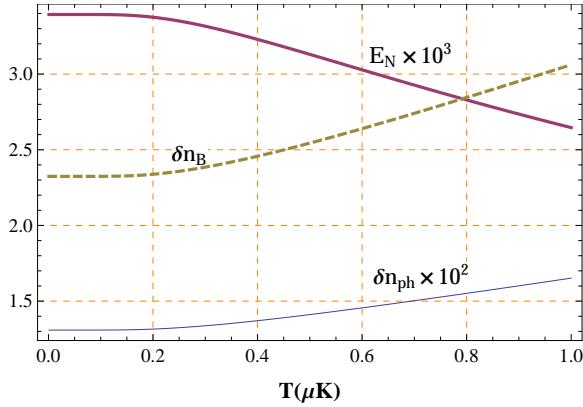


FIG. 5. (color online) The effects of temperature on the incoherent excitation numbers of photons (thin line) and atoms in the Bogoliubov side mode (dashed line) as well as the entanglement between photons and atoms (thick line). The cavity detuning has been set at $\Delta_c = 28900\omega_R$ and the s-wave scattering frequency has been considered to be $\omega_{sw} = \omega_R$. The other parameters are the same as those of Fig.2. For clarity, the values of δn_{ph} and E_N have been multiplied by 100 and 1000 respectively.

resonance but they increase rapidly near the resonance. On the other hand increasing atom-atom interaction decreases the fluctuations in the number of photons and atoms. Since the total number of photons is the mean value α^2 that obtained in previous subsection plus the incoherent excitation number of photons, we can conclude that the stronger atom-atom interaction, the weaker optical field of the cavity. In fact the atom-atom interaction makes larger shifts in the cavity resonance frequency and consequently reduces the cavity field intensity. Hence the decrease of the cavity output provides a direct measure of the atom-atom interaction within the condensate.

in Fig.4 (c) the entanglement between the Bogoliubov mode c_{10} and the optical field has been plotted versus the normalized cavity detuning Δ_c/ω_R for two values of ω_{sw} . Again, the maximum of entanglement occurs at resonance. Besides, an increasing in the s-wave scattering frequency causes the entanglement of the atoms and photons to decrease. As it was mentioned above, an increase in atom-atom interaction shifts the resonance of the cavity which reduces the number of photons of the cavity and also causes the depletion of the Bogoliubov mode. This reduction in the mean values of photons and atoms leads to a reduction in the effective atom-photon coupling (Fig.3(b)) which causes the entanglement of atoms and photons to decrease.

The thermal effects on the incoherent excitation numbers of photons and atoms in the Bogoliubov side mode as well as the entanglement between photons and atoms have been illustrated in Fig.5. As is seen from the figure, increasing the temperature causes the fluctuations of the number of photons and atoms to increase and on the other hand reduces the atom-photon entanglement. Furthermore, the thermal effects for the range of temper-

atures below $0.1\mu K$ are negligible.

V. SUSCEPTIBILITY AND POWER SPECTRUM OF THE BOGOLIUBOV MODE

Finally, we are going to obtain the power spectrum of the Bogoliubov side mode c_{10} and also drive its effective frequency and damping rate. We will show that the coupling between the cavity field and the Bogoliubov mode which behaves like a mechanical mirror, leads to the splitting of the normal mode into two modes (NMS). A similar theoretical approach has been done in Ref.[36] for optomechanical Bose-Hubbard Hamiltonian (OMBH) by expanding the atomic wave field in terms of the Wannier functions which is valid only for weak atom-field nonlinearity [37]. Instead, here we use the atomic field expansion, Eq.(12), in the momentum space. For this purpose, we solve the linearized QLEs for the fluctuations in the displacement operator δX_{10} as

$$\delta X_{10}(\omega) = \chi(\omega) F_T(\omega), \quad (35)$$

where $F_T(\omega)$ is the Fourier transformation of the fluctuations in the total force acting on the Bogoliubov mode and $\chi(\omega)$ is its susceptibility which are given by, respectively,

$$F_T(\omega) = -\frac{G(\kappa - i\omega)}{\Delta_d^2 + (\kappa - i\omega)^2} \xi_x(\omega) + \frac{G\Delta_d}{\Delta_d^2 + (\kappa - i\omega)^2} \xi_p(\omega) + \frac{\gamma - i\omega}{\Omega_{10}^{(-)}} f_{x10}(\omega) + f_{p10}(\omega), \quad (36a)$$

$$\chi(\omega) = \frac{\Omega_{10}^{(-)}}{\Omega_{eff}^2 - \omega^2 - i\omega\Gamma_{eff}}, \quad (36b)$$

where Ω_{eff} is the effective frequency of the Bogoliubov mode,

$$\Omega_{eff}^2 = \gamma^2 + \omega_m^2 + \frac{\Delta_d G^2 \Omega_{10}^{(-)} (\Delta_d^2 + \kappa^2 - \omega^2)}{(\Delta_d^2 + \kappa^2 - \omega^2)^2 + 4\omega^2\kappa^2}, \quad (37)$$

and Γ_{eff} is its effective damping rate,

$$\Gamma_{eff} = 2\gamma - \frac{2\kappa\Delta_d G^2 \Omega_{10}^{(-)}}{(\Delta_d^2 + \kappa^2 - \omega^2)^2 + 4\omega^2\kappa^2}. \quad (38)$$

To calculate the spectrum, we need the correlation functions of the optical noise sources in the frequency domain

$$\langle \xi_x(\omega) \xi_x(\omega') \rangle = \langle \xi_p(\omega) \xi_p(\omega') \rangle = \kappa \delta(\omega + \omega'), \quad (39a)$$

$$\langle \xi_x(\omega) \xi_p(\omega') \rangle = \langle \xi_p(\omega) \xi_x(\omega') \rangle^* = i\kappa \delta(\omega + \omega'), \quad (39b)$$

where we have assumed $n_{ph} = 0$. For the atomic noise sources in the frequency domain we have

$$\langle f_x(\omega) f_x(\omega') \rangle = 2\gamma(n_B + \frac{1}{2}) \delta(\omega + \omega'), \quad (40a)$$

$$\langle f_p(\omega) f_p(\omega') \rangle = 2\gamma(n_B + \frac{1}{2}) \delta(\omega + \omega'), \quad (40b)$$

$$\langle f_x(\omega) f_p(\omega') \rangle = i\gamma \delta(\omega + \omega'), \quad (40c)$$

$$\langle f_p(\omega) f_x(\omega') \rangle = -i\gamma \delta(\omega + \omega'), \quad (40d)$$

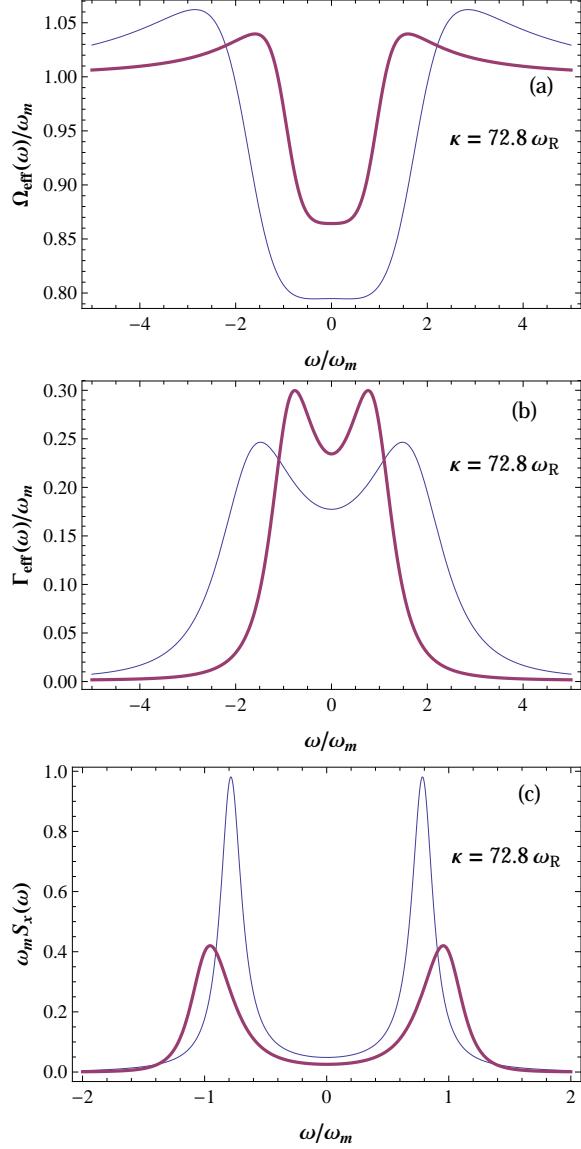


FIG. 6. (Color online) (a) Normalized effective Bogoliubov mechanical frequency Ω_{eff}/ω_m , (b) normalized effective Bogoliubov mechanical damping Γ_{eff}/ω_m and (c) normalized power spectrum of the displacement operator of the Bogoliubov mode versus the normalized frequency ω/ω_m for two values of $\omega_{sw} = 80\omega_R$ (thin line) and $\omega_{sw} = 140\omega_R$ (thick line) and for $\Delta_c = 28700\omega_R$ and $\kappa = 72.8\omega_R$. The other parameters are the same as those of Fig.2.

where we have omitted the indices of f_x and f_p for simplicity. Here, $n_B = [\exp(\frac{-\hbar\omega_m}{k_B T}) - 1]^{-1}$ is the number of thermal excitations for the Bogoliubov mode and ω_m is the mechanical Bogoliubov frequency of oscillation given by Eq.(27). By using the relation $S_x(\omega) = \frac{1}{4\pi} \int d\omega' e^{-i(\omega+\omega')t} \langle \delta X_{10}(\omega) \delta X_{10}(\omega') + \delta X_{10}(\omega') \delta X_{10}(\omega) \rangle$ we can calculate the power spectrum of the displacement

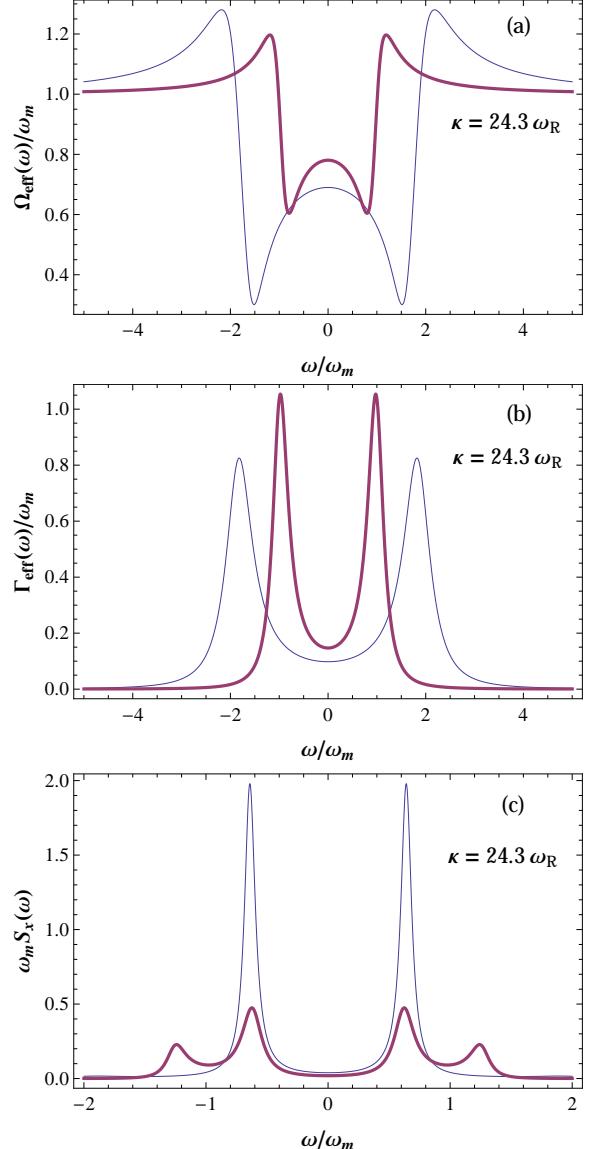


FIG. 7. (Color online) (a) Normalized effective Bogoliubov mechanical frequency Ω_{eff}/ω_m , (b) normalized effective Bogoliubov mechanical damping Γ_{eff}/ω_m and (c) normalized power spectrum of the displacement operator of the Bogoliubov mode versus the normalized frequency ω/ω_m for two values of $\omega_{sw} = 80\omega_R$ (thin line) and $\omega_{sw} = 140\omega_R$ (thick line) and for $\Delta_c = 28700\omega_R$ and $\kappa = 24.3\omega_R$. The other parameters are the same as those of Fig.2.

operator $\delta X_{10}(\omega)$ as follows

$$S_x(\omega) = \frac{1}{4\pi} |\chi(\omega)|^2 \left[4\gamma(n_B + \frac{1}{2}) \frac{\gamma^2 + \omega^2 + \Omega_{10}^{(-)2}}{\Omega_{10}^{(-)2}} + \frac{2\kappa G^2 (\Delta_d^2 + \omega^2 + \kappa^2)}{(\Delta_d^2 + \kappa^2 - \omega^2)^2 + 4\omega^2 \kappa^2} \right]. \quad (41)$$

The modification of the frequency of the Bogoliubov excitations of the condensate due to the radiation pressure given by Eq.(37) is equivalent to the optical spring effect

in cavity optomechanical systems with movable mirrors [1].

In Figs. 6(a) and 7(a) we have plotted the normalized effective Bogoliubov mechanical frequency Ω_{eff}/ω_m versus the normalized frequency ω/ω_m for two values of cavity damping rates $\kappa = 72.8\omega_R$ (Fig.6 (a)) and $\kappa = 24.3\omega_R$ (Fig.7 (a)), and for two different values of $\omega_{sw} = 80\omega_R$ (thin line) and $\omega_{sw} = 140\omega_R$ (thick line). As is seen from both figures, a higher two-body interaction makes the condensate more robust and the Bogoliubov frequency of the condensate does not significantly deviate from ω_m . Besides, a decrease in the damping rate of cavity causes the appearance of the peaks in the curves.

In Figs.6(b) and 7(b) the normalized effective Bogoliubov mechanical damping Γ_{eff}/ω_m has been plotted versus the normalized frequency ω/ω_m for two values of cavity damping rates $\kappa = 72.8\omega_R$ (Fig.6 (b)) and $\kappa = 24.3\omega_R$ (Fig.7 (b)). In the case of Fig.6(b) where the damping rate of the cavity is nearly equal to the effective atom-photon coupling constant ($\kappa \simeq G$), the effective damping rate Γ_{eff} is much lower compared to the case of Fig.7(b) where $G > \kappa$. It means that the stronger atom-photon coupling, the higher atom loss and hence the higher value of the effective damping. An experimental observation of this phenomenon of light induced back-action heating and consequent loss of atoms has been reported in Ref.[38]. It was also found that the atom loss rate is increased near the resonance. On the other hand, the effect of atomic collisions on the effective damping rate has been demonstrated in Figs.6(b) and 7(b) which show that a stronger atom-atom interaction causes the effective damping rate to be increased near the resonance points. Therefore increasing the rate of atomic collisions can help us in cooling of the Bogoliubov mode of the BEC by the radiation pressure.

Finally, we are going to investigate the influence of atomic collisions on the phenomenon of NMS in the power spectrum of Bogoliubov displacement operator. The NMS is associated with the mixing between the fluctuation of the cavity field around the steady state and the fluctuations of the condensate (Bogoliubov mode) around the mean field. The origin of the fluctuations of the cavity field is the beat of the pump photons with the photons scattered from the condensate atoms. The phenomenon of NMS is observable whenever the energy exchange between the two interacting modes takes place on a time scale faster than the decoherence of each mode.

In Figs.6(c) and 7(c) we have shown the normalized power spectrum of the displacement operator of the Bogoliubov mode S_X versus the normalized frequency ω/ω_m for two values of cavity damping rate $\kappa = 72.8\omega_R$ (Fig.6 (c)) and $\kappa = 24.3\omega_R$ (Fig.7 (c)), and for two values of $\omega_{sw} = 80\omega_R$ (thin line) and $\omega_{sw} = 140\omega_R$ (thick line). In the case of Fig.6(c) where the damping rate of the cavity is nearly equal to the effective coupling constant of atoms and photons ($\kappa \simeq G$), there is no splitting in the power spectrum (neither for $\omega_{sw} = 80\omega_R$ nor for $\omega_{sw} = 140\omega_R$) while in Fig.7(c) where $G > \kappa$ the NMS is appeared for

$\omega_{sw} = 140\omega_R$. So in order to observe NMS firstly the system should be in the strong coupling regime where the atom-photon coupling is larger than the decay rates of photons and atoms. When this condition is fulfilled, the NMS can be observable with increasing atom-atom interaction. Normal mode splitting of a system of large number of atoms coupled to a ring cavity has been observed experimentally in the strong cooperative coupling regime [39]. The splitting of the normal mode has been observed by increasing the number of atoms which leads to the increase in the s-wave scattering frequency.

VI. CONCLUSIONS

In conclusion, we have done a theoretical investigation on the optomechanical properties of a one-dimensional Bose-Einstein condensate inside a driven optical cavity considering the effects of atomic collisions. Due to the dispersive atom-photon interaction the atoms develop a band structure in the optical lattice of the cavity. In the limit of weak photon-atom coupling the lowest bands $n = \pm 1$ can be excited by fluctuations due to the atom-light interaction. On the other hand, the atomic collisions scatter the atoms to the higher band and also populate states with non zero quasimomentum. It has been shown that there is a nonlinear optomechanical coupling between the nonzero quasimomentum states with the optical field proportional to $1/\sqrt{N}$ which can be disregarded in the Bogoliubov approximation where the number of atoms in the BEC mode is very large. Therefore, there is no radiation pressure coupling between the atomic modes with nonzero quasimomentum and the cavity field in the Bogoliubov approximation.

In this way we have obtained a simplified optomechanical model where the Bogoliubov side mode c_{10} is coupled to the optical field through the radiation pressure term and the atom-atom interaction Hamiltonian behaves like an OPA for the Bogoliubov mode with the non-linear gain parameter ω_{sw} . It has been shown that in the strong coupling regime where the effective atom-photon coupling is greater than the cavity damping rate, an increase in the s-wave scattering frequency would lead to the NMS in the power spectrum of the Bogoliubov mode.

Furthermore, the atom-atom interaction causes the depletion of the Bogoliubov mode and also shifts the cavity resonance frequency which leads to a decrease in the mean number of cavity photons. Besides, it decreases fluctuations in the number of photons and atoms. Hence the decrease in the cavity output provides a direct measure of the atom-atom interaction within the condensate. On the other hand, a stronger atomic collision rate causes a decrease in the effective atom-photon coupling which leads to a decrease in the entanglement between the Bogoliubov mode and the optical field.

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